Pergamon

J. Appl. Maths Mechs, Vol. 58, No. 3, pp. 531–537, 1994

0021–8928/94 $24.00+0.00

0021-8928(94)00077-8

APPLICATION OF THE THEORIES OF REPRESENTATIONS TO THE PROBLEMS OF THE EXCITATION AND REFLECTION OF WEDGE WAVES†

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(*Received* 3 *September* 1993)

A method of solving the problem of the reflection of a wedge wave at the edge of an acute-angled elastic wedge, that is invariant under a two-parametric group of transformations, is proposed. Representations of this group are used to obtain the mean values of the intensities of the fictitious sources situated at the edge of the wedge. The mean values obtained are used to construct the unknown functions using an expansion in Hermite polynomials.

The difficulty in solving the problem of the excitation of wedge waves from the edge of an acute-angled wedge, and also the problem of the reflection of a wave incident at the end, is due to the following. First, the problem of scattering at the end has no small parameters as, for example, in the problem of the scattering by a defect of the edge [1], and hence it cannot be solved by perturbation methods. Second, the eigenfunction of the problem of the propagation of wedge modes in an acute-angled wedge are expressed in terms of special functions [2], and it is not possible to use them to construct Fourier expansions. Third, the presence of a pair of boundary conditions does not enable the reflection method to be applied to the problem of scattering at the end.

The problem of the excitation of a wedge wave has a two-parameter group of symmetries. As we know, the presence of a group of symmetries enables the order of the ordinary differential equations [3] to be reduced, and enables the variables to be separated in the partial differential equations [4]. In this paper representations of the group of symmetries are used to construct a generalized Fourier transformation for the integral equation, to which the problem in question can be reduced by means of potential theory.

1. FORMULATION OF THE PROBLEM

Consider the problem of the excitation of oscillations in an acute-angled elastic wedge. Suppose the wedge has an end to which concentrated forces and moments are applied. Suppose the edge of the wedge lies along the y axis while the end lies along the x axis. The wedge therefore occupies the region $x>0$, $y>0$.

We will use the theory of thin plates [2] to describe the oscillations of the acute-angled elastic wedge. The equation for the transverse antisymmetric oscillations has the form

†*Prikl. Mat. Mekh.* Vol. 58, No. 3, pp. 154–160, 1994.

$$\frac{\partial^2 M_x}{\partial x^2}+\frac{\partial^2 M_y}{\partial y^2}-2\frac{\partial^2 M_{xy}}{\partial x\partial y}-mx\frac{\partial^2 w}{\partial t^2}=\frac{I(x,y,t)}{x} \tag{1.1}$$

$$M_x=-D_0x^3\left(\frac{\partial^2 w}{\partial x^2}+\nu\frac{\partial^2 w}{\partial y^2}\right),\ M_y=-D_0x^3\left(\frac{\partial^2 w}{\partial y^2}+\nu\frac{\partial^2 w}{\partial x^2}\right)$$

$$M_{xy}=D_0(1-\nu)x^3\frac{\partial^2 w}{\partial x\partial y},\ D_0=\frac{2E}{3(1-\nu^2)}\ \mathrm{tg}^3\left(\frac{\theta}{2}\right),\ m=2\rho\,\mathrm{tg}\left(\frac{\theta}{2}\right)$$

where $w(x,y,t)$ is the transverse displacement of points of the wedge, E and ν are the elastic constants, ρ is the density of the material, θ is the aperture angle of the wedge, and $I(x, y, t)$ is an auxiliary function representing fictitious sources and which is equal to zero in the region occupied by the wedge.

The following boundary conditions are satisfied on the edge of the wedge (where $x=0$)

$$M_x=0,\ \partial M_x/\partial x-2\partial M_{xy}/\partial y=0 \tag{1.2}$$

which in our problem reduces to the requirement that w must be finite in the vicinity of the edge. We define the following operators at the end of the wedge

$$\Gamma_1[w]\equiv x^{-1}M_y\big|_{y=0}=f_1(x,t)$$

$$\Gamma_2[w]\equiv(\partial M_y/\partial y-2\partial M_{xy}/\partial x)\big|_{y=0}=f_2(x,t) \tag{1.3}$$

Here the functions $xf_1(x, t)$ and $f_2(x, t)$ are the concentrated moment and force applied to the end of the wedge. Note that the factor x in front of the moment f_1 is chosen to ensure algebraic homogeneity of the zeroth degree in both boundary conditions.

The problem of the reflection of a wedge wave reduces to the problem of excitation if we put the functions f_1 and f_2 on the right-hand side of (1.3) equal to $-f_1'$ and $-f_2'$, respectively, where f_1' and f_2' are the values of the operators Γ_1 and Γ_2 for the incident wave.

We will use potential theory to solve the boundary-value problem (1.1)–(1.3). We will introduce an inhomogeneity I at the end of the wedge in the form of a sum of simple-layer and double-layer type sources and we will require that the solution of Eq. (1.1) obtained in the region $(-\infty<y<\infty)$ should satisfy the boundary conditions (1.3) when $y=+0$. We will choose the algebraically homogeneous function I in the form

$$I(x,y,t)=\varphi_1(x,t)x^2\delta'(y)+\varphi_2(x,t)x\delta(y) \tag{1.4}$$

where δ is the Dirac delta function, δ' is its derivative, and φ_i are weighting factors.

To solve Eq. (1.1), (1.4) we will introduce Green's function $G(x,y,t,x',y',t')$, which is the solution of Eq. (1.1) when $I(x, y, t)=\delta(x-x')\delta(y-y')\delta(t-t')$. The function G must satisfy the radiation conditions. A procedure for calculating G is given in the Appendix. The solution of Eq. (1.1) is the convolution of Green's function with (1.4)

$$w(x,y,t)=\int_{-\infty}^{\infty}\int_{0}^{\infty}\int G(x,y,t,x',y',t')I(x',y',t')dx'dy'dt' \tag{1.5}$$

The values of the operators $\Gamma_i[\omega]$ have discontinuities at $y=0$, equal to respectively [5]

$$\Gamma_i[\omega]\big|_{y=+0}-\Gamma_i[\omega]\big|_{y=-0}=\varphi_i(x,t) \tag{1.6}$$

Bearing in mind that the Fourier representation of the derivatives of Green's function (A.1) gives arithmetic mean values on the right and left at the point of discontinuity and that the boundary conditions are being sought at $y=+0$, we obtain

$$\Gamma_i[\omega]\Big|_{y=+0} = \tfrac{1}{2}\varphi_i + \Gamma_i[\omega]\Big|_{y=0} \tag{1.7}$$

Substituting (1.5) into (1.7) and using the evenness of G along the y axis, we finally obtain

$$f_i(x,t) = \int_{-\infty}^{\infty}\int_{0}^{\infty} G_{ij}^{*}(x,t,x',t')\varphi_j(x',t')dx'dt' \tag{1.8}$$

Here and henceforth summation is taken over repeated indices and

$$G_{11}^{*} = G_{22}^{*} = \tfrac{1}{2}\delta(x-x')\delta(t-t'),\ \ G_{12}^{*} = \Gamma_1[xG(x,y,t,x',0,t')]$$

$$G_{21}^{*} = \Gamma_2[x^2\frac{\partial}{\partial y}G(x,y,t,x',0,t')] \tag{1.9}$$

The operators Γ_i act on the variables x and y.

Hence, the problem of the excitation of a wedge wave reduces to integral equation (1.8) in φ_i.

2. SYMMETRY GROUP

The coordinates x and y will henceforth be assumed to be dimensionless. It can be established by a direct check that Eq. (1.8) admits of a two-parameter group H of transformations of the coordinates, i.e. it remains valid when the transformation $h \in H$ is applied simultaneously to the arguments of the functions φ_i and f_i.

The group H is generated by shifts in time and homothety of the half-plane $x > 0$ with arbitrary positive coefficients. We will parameterize the group H as follows. Suppose that as a result of the action of an element of the group $h(x_0,\ t_0)$ the point $(x,\ t)$ is transferred to the point

$$h(x_0,t_0)\circ(x,t) = (xx_0, tx_0 + t_0) \tag{2.1}$$

Group operation is specified by the formula

$$h(x_2,t_2)h(x_1,t_1) = h(x_2x_1, x_2t_1 + t_2) \tag{2.2}$$

We will introduce into the group H the left-invariant measure

$$d\mu_L(h(x,t)) = x^{-2}dxdt \tag{2.3}$$

It was obvious that the kernel $\|G_{ij}^{*}\|$ can be represented using a function of two arguments, invariant under the transformations of the group H

$$G_{ij}^{*}(x,t,x',t') = (x')^{-2}G_{ij}(x/x',\ (t-t')/x') \tag{2.4}$$

Since the parameters x_0 and t_0 of the group H are defined in the same region as the coordinates x and t, we can assume that all the functions are defined on the elements of the group H. By (2.3) and (2.4)

$$F_i(h_o) = \int_H G_{ij}(h_r^{-1}h_o)\varphi_j(h_r)d\mu_L(h_r) \tag{2.5}$$

where h_0 and h_r are the elements of the group corresponding to the points of observation and radiation, respectively.

Note that the right-hand side of (2.5) is the convolution of $\|G_{ij}\|$ and $\{\varphi_j\}$ on the group H [6]. Consequently, a generalized Fourier transformation can be used to solve Eq. (2.5). To do this we introduce a two-parameter family of representations $T(h)$ of the group H. Eq. (2.5) is averaged over the group H with weighting factors defined by each of the representations T, and we are then able to express the mean of φ_i in terms of the mean of f_i. The functions $\varphi_i(h)$ are then synthesized from the means of φ_i over all the representations.

Suppose T is an n-dimensional representation of the group H. We will define Φ_i and F_i as column-vectors in n-dimensional space of the representation T, where one of the coordinates of Φ_i and F_i in a certain basis is equal to $\varphi_i(h)$ and $f_i(h)$ respectively, while the remaining $n-1$ coordinates are equal to zero. Note that the operator $\|G_{ij}\|$ is scalar in this space.

Using the commutivity of the operator $\|G_{ij}\|$ with any matrix, which is independent of h_r, we can write the following expressions for the means of

$$\langle T,F_i\rangle_k = \int\limits_H T_{kl}(h_o^{-1})\int\limits_H G_{ij}(h_r^{-1}h_o)\Phi_{jl}(h_r)d\mu_L(h_r)d\mu_L(h_o) =$$

$$= \iint\limits_H G_{ij}(h_r^{-1}h_o)T_{km}(h_o^{-1}h_r)T_{ml}(h_r^{-1})\Phi_{jl}(h_r)d\mu_L(h_r^{-1}h_o)d\mu_L(h_r) =$$

$$= \int\limits_H G_{ij}(\tau)T_{kl}(\tau^{-1})\langle T,\Phi_j\rangle_l d\mu_L(\tau),\quad \tau = h_r^{-1}h_o \tag{2.6}$$

The means of $\langle T, F_i\rangle_k$ and $\langle T,\Phi_i\rangle_k$ are defined by the expressions

$$\langle T,F_i\rangle_k \equiv \int\limits_H T_{kl}(h^{-1})F_{il}(h)d\mu_L(h),\ \langle T,\Phi_i\rangle_k \equiv \int\limits_H T_{kl}(h^{-1})\Phi_{il}(h)d\mu_L(h)$$

where T_{kl} is the matrix of the representation T and the subscripts k and l indicate the components in representation space.

Note that the vector $\langle T,\Phi_i\rangle$ is independent of the coordinates. Its components can be determined as follows. We defined the matrices $\|G_{ijkl}\|$ and $\|G_{ijkl}\|^{-1}$ as

$$G_{ijkl}\int\limits_H G_{ij}(\tau)T_{kl}(\tau^{-1})d\mu_L(\tau),\quad G^{-1}_{ijkl}G_{j\upsilon lm} = \delta_{i\upsilon}\delta_{km} \tag{2.7}$$

Then

$$\langle T,F_i\rangle_k = G_{ijkl}\langle T,\Phi_j\rangle_l,\ \langle T,\Phi_i\rangle_k = G^{-1}_{ijkl}\langle T,F_j\rangle_l \tag{2.8}$$

in the case of a non-degenerate matrix $\|G_{ijkl}\|$.

3. THE CONSTRUCTION OF REPRESENTATIONS OF H AND THE SYNTHESIS OF THE SOLUTION

The group H has no two-parameter family of one-dimensional representations [7]. This family would represent the possibility of a direct construction of the Fourier expansion. This group H admits of a two-parameter family of reducible unexpanded representations $T^{\alpha,n}$, where α is a continuous complex parameter, while $n = 0, 1, 2, \ldots$ The dimensions of each of the representations $T^{\alpha,n}$, is $n+1$.

The basis of the representation $T^{\alpha,n}$ is the set of functions

$$\mathbf{e}_k = t^k x^\alpha,\quad k = 0,1,\ldots,n,\quad \alpha \in \mathrm{C} \tag{3.1}$$

defined on the half-plane $x > 0$. We will obtain the matrix $T^{\alpha,n}_{kl}$ which has the property

$$h(x_0,t_0)\circ\mathbf{e}_l = T^{\alpha,n}_{kl}\mathbf{e}_k$$

Analysing the action of the elements of the group H on the function (3.1) and using a binomial expansion we obtain

$$T_{kl}^{\alpha,n}(h(x,t)) = (-1)^{l-k} C_l^k x^{-\alpha+l} t^{l-k}, \quad T_{kl}^{\alpha,n}(h^{-1}(x,t)) = C_l^k x^{\alpha+k} t^{l-k} \tag{3.2}$$

When $l<k$ the corresponding value of C_l^k is assumed to be equal to zero.

We will define the vectors Φ_{ik} and F_i as

$$\Phi_{ik} = \delta_{kn}\varphi_i, \quad F_{ik} = \delta_{kn} f_i, \quad k = 0,1,\dots,n \tag{3.3}$$

i.e. the last coordinates are non-zero. The components $\langle T, \Phi_i\rangle$, $\langle T, F_i\rangle$ and $\| G_{ijkl} \|$ can be expressed by the following formulae

$$\langle T, \Phi_i\rangle_k = C_n^k \int_0^\infty \int_{-\infty}^\infty x^{\alpha+k} t^{n-k} \varphi_i(x,t) x^{-2} dt dx \tag{3.4}$$

$$\langle T, F_i\rangle_k = C_n^k \int_0^\infty \int_{-\infty}^\infty x^{\alpha+k} t^{n-k} F_i(x,t) x^{-2} dt dx \tag{3.5}$$

$$G_{ijkl} = C_l^k \int_0^\infty \int_{-\infty}^\infty x^{\alpha+k} t^{l-k} G_{ij}^*(x,t,1,0) x^{-2} dt dx \tag{3.6}$$

Here $\langle T, F_i\rangle$ is calculated from the boundary conditions of problem (1.1)–(1.3), while $\langle T, \Phi_i\rangle$ is found from (2.8).

The integrals in (3.4)–(3.6) must be understood as action on the functions φ_i, F_i and G_{ij}^* of the generalized functions $x^\alpha t^n$, regularized as, for example, in [8].

The functions $\varphi_i(x, t)$ are synthesized as follows. Put $k=0$ and $\alpha=-i\omega+1$. The integrals over x are a Fourier transformation with respect to the variable $\ln x$ and can be expressed in the usual way. An expansion in the variable t can be constructed in a system of orthogonal polynomials, for example, Hermite polynomials. Suppose H_n^m are the coefficients of orthonormalized Hermite polynomials, i.e. [9]

$$H_n(t) = \sum_0^n H_n^m t^m, \quad \int_{-\infty}^\infty e^{-t^2} H_m(t) H_n(t) dt = \delta_{mn} \tag{3.7}$$

It is obvious that

$$\int_{-\infty}^\infty \int_0^\infty H_n(t)\varphi_i(x,t) x^{\alpha-2} dx dt = \sum_{m=0}^n H_n^m \langle T^{\alpha,m}, \Phi_i\rangle_0 \tag{3.8}$$

Formulae of a Fourier transformation with respect to the variable $\ln x$, and also the relations of orthogonality of Hermite polynomials (3.7) gives the following expansion

$$\varphi_i = \frac{e^{-t^2}}{2\pi} \int_{-\infty}^\infty x^{i\omega} \sum_{n=0}^\infty H_n(t) \sum_{m=0}^n H_n^m \langle T^{\alpha,m}, \Phi_i\rangle_0 d\omega \tag{3.9}$$

Equation (3.9), where $\langle T, \Phi_i\rangle$ are found from (2.8), is the solution of Eq. (1.8).

4. DISCUSSION OF THE RESULTS

We have proposed a method of solving the problem of the radiation of wedge acoustic waves, the essence of which reduces to the following.

Green's function $G(x, y, t\ x', y', t')$ is calculated (see the Appendix). The function G_{ij} is constructed from (1.9). Using (3.6) G_{ijkl} is calculated for all real ω and non-negative integers k and l. The matrix $\|G_{ijkl}\|$ is inverted.

The functions f_i are synthesized using (1.3). The means $\langle T, F_i \rangle$ are calculated from (3.5) and $\langle T, \Phi_i \rangle$ from (2.8) using $\|G_{ijkl}\|^{-1}$.

The functions $\varphi_i(x, t)$ are obtained from (3.9). The solution is found by the convolution (1.5) and (1.4).

The technique for evaluating the integrals and series (3.4)–(3.9) requires further development.

APPENDIX

We will obtain Green's function of problem (1.1) and (1.2). Bearing in mind that $G = G(x, y-y', t-t', x')$ and assuming $y' = 0$ and $t' = 0$, we can write

$$G(x,y,t,x',0,0) = \int\int_{-\infty}^{\infty} \overline{G}_{k,\omega}(x,x')e^{i(ky+\omega t)}dkd\omega \tag{A.1}$$

Equation (1.1) reduces to the following ordinary differential equation (x' is a parameter)

$$\frac{d^2}{dx^2}\left[-D_0x^3\left(\frac{d^2\overline{G}}{dx^2} - \nu k^2\overline{G}\right)\right] + k^2x^3D_0\left(-k^2\overline{G} + \nu\frac{d^2\overline{G}}{dx^2}\right) +$$

$$+2D_0(1-\nu)k^2\frac{d}{dx}\left(x^3\frac{d\overline{G}}{dx}\right) + mx\omega^2\overline{G} = \frac{1}{(2\pi)^2x'}\delta(x-x') \tag{A.2}$$

In the region $0 < x < x'$ there are two solutions bounded as $x \to 0$ [2, 9]

$$w_1 = M(-\xi,2;\ 2kx)e^{-kx},\quad w_2 = M(2+\xi,2;\ 2kx)e^{-kx}$$

$$\xi = -1 + \tfrac{1}{2}[6\nu - 2 + 3(1-\nu^2)\rho E^{-1}\mathrm{tg}^{-2}(\theta/2)(\omega/k)^2]^{1/2}$$

Here $M(a, b; z)$ is the Kummer function. In the region $x > x'$ there are two solutions that are bounded as $x \to \infty$

$$w_3 = U(-\xi,2;\ 2kx)e^{-kx},\quad w_2 = U(2+\xi,2;\ 2kx)$$

where $U(a, b; z)$ is the confluent hypergeometric function.

The solution of Eq. (A.2) is

$$\overline{G}(x) = -(a_1w_1 + a_2w_2),\ 0 < x < x';\ \overline{G}(x) = (a_3w_3 + a_4w_4),\ x > x' \tag{A.3}$$

where a_i are constants, found from the equations

$$\sum_{i=1}^{4} a_iw_i^{(j-1)} = -\delta_{j4}D_0^{-1}(2\pi)^{-2}(x')^{-4},\ j = 1,\dots 4$$

Suppose W_{kl} is the inverse matrix to $w_i^{(j-1)}$. Then $a_i(x') = -D_0^{-1}(2\pi)^{-2}(x')^{-4}W_{i4}(x')$.

Reverting to (A.3) and (A.1), we obtain the required representation of $\overline{G}$.

This representation of $\overline{G}$ is not unique. The representation of $\overline{G}$ in the form of a series in Laguerre polynomials may turn out to be more convenient. Suppose $L[w]$ is the operator on the left-hand side of (A.2). The following equations hold [12]

$$L[L_n^1(2k_nx)e^{-k_nx}] = 0,\quad k_n = \omega/c_n$$

where L_m^1 are Laguerre polynomials of the first kind, while c_m is the velocity of the wedge waves. It is obvious that for any w and k

$$L[L_n^1(2k_n x)e^{-k_n x}] = mx(\omega^2 - c_n^2 k^2)L_n^1(2kx)e^{-kx}$$

Using the orthogonality relation for Laguerre polynomials [9]

$$\int_0^\infty e^{-z} z L_m^1(z) L_n^1(z)dz = (n+1)\delta_{mn}$$

we can represent the function $\overline{G}$ in the form of a series

$$\overline{G}_{k,\omega}(x,x') = \sum_{n=0}^{\infty} \frac{k^2 e^{-k(x'+x)} L_n^1(2kx') L_n^1(2kx)}{mn^2 x'(n+1)(\omega^2 - c_n^2 k^2)}$$

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Translated by R.C.G.